LAPLACE OPERATOR

The Laplace operator in the spatial domain

The Laplace operator is defined by: $$\nabla^2 f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \quad (1)$$

In the discrete case we get $$\nabla^2 f(i,j) \equiv \Delta_x^2 f(i,j) + \Delta_y^2 f(i,j) \quad (2)$$

where

$$\begin{aligned}
\Delta_x f(i,j) &\equiv f(i,j) - f(i-1,j) \\
\Delta_y f(i,j) &\equiv f(i,j) - f(i,j-1) \\
\Delta_x^2 f(i,j) &\equiv \Delta_x f(i+1,j) - \Delta_x f(i,j) \\
&\equiv [f(i+1,j) - f(i,j)] - [f(i,j) - f(i-1,j)] \\
&\equiv f(i+1,j) + f(i-1,j) - 2f(i,j) \\
\Delta_y^2 f(i,j) &\equiv f(i,j+1) + f(i,j-1) - 2f(i,j)
\end{aligned}$$

It follows that:

$$\nabla^2 f \equiv [f(i+1,j) + f(i-1,j) + f(i,j+1) + f(i,j-1)] - 4f(i,j) \quad (3)$$

Notice that this result is proportional to

$$f(i,j) - \tfrac{1}{5}[f(i+1,j) + f(i-1,j) + f(i,j) + f(i,j+1) + f(i,j-1)] \quad (4)$$

Hence, the discrete Laplace operator can be replaced by the original function subtracted by an average of this function in a small neighborhood:

$$\nabla^2 f = f(i,j) - \bar{f}(i,j) \quad (5)$$

Laplace operator in the frequency domain

$$f(X_1, X_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} U_0(X_1 - m\Delta) U_0(X_2 - n\Delta) \quad (6)$$

where $$U_0 = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$$ (Kronecker delta)

$$
\begin{aligned}
F(jU_1, jU_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} U_0(X_1 - m\Delta) U_0(X_2 - n\Delta) e^{-j(U_1X_1 + U_2X_2)} dX_1 dX_2 \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} \int_{-\infty}^{\infty} U_0(X_1 - m\Delta) e^{-jU_1X_1} dX_1 \int_{-\infty}^{\infty} U_0(X_2 - n\Delta) e^{-jU_2X_2} dX_2 \\
&= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} e^{-j(U_1 m\Delta + U_2 n\Delta)}
\end{aligned}
$$

$$
\left(\int_{-\infty}^{\infty} f(X) U_0(X - X') dX = f(X') \right)
$$

Laplace operator

The Laplace operator is defined by:

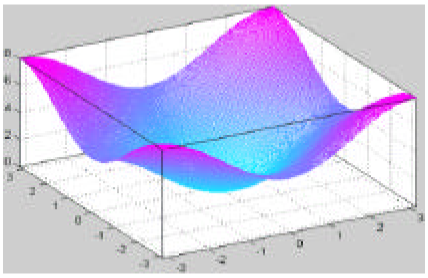
$$
\begin{array}{lll}
a_{-1,1} = 0 & a_{0,1} = -1 & a_{1,1} = 0 \\
a_{-1,0} = -1 & a_{0,0} = 4 & a_{1,0} = -1 \\
a_{-1,-1} = 0 & a_{0,-1} = -1 & a_{1,-1} = 0
\end{array}
$$

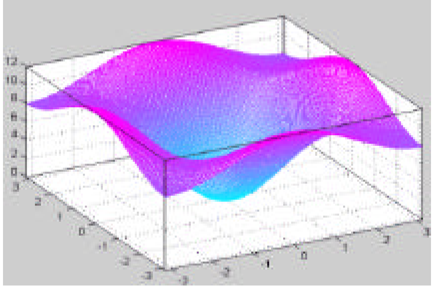
Using the expression above we get:

$$
\begin{aligned}
F(jU_1, jU_2) &= -e^{j\Delta U_2} - e^{j\Delta U_1} + 4 - e^{-j\Delta U_2} - e^{-j\Delta U_1} \\
&= 4 - 2\cos \Delta U_1 - 2\cos \Delta U_2 \\
&= 2(2 - \cos \Delta U_1 - \cos \Delta U_2)
\end{aligned}
$$

Below you can see the magnitude of the Fourier transform of two different discrete approximations of the Laplacian. Notice, both are variant to rotations.

$$
\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}
\qquad
\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}
$$



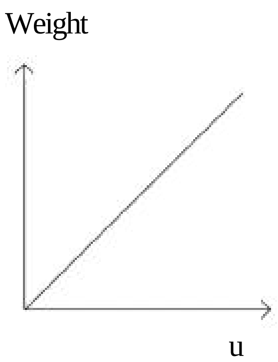


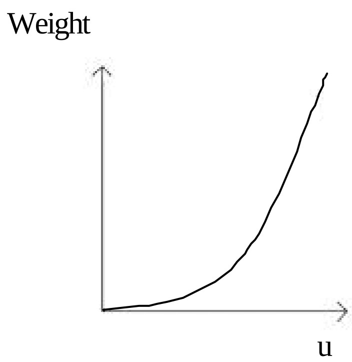
One drawback with the Laplacian is that it is sensitive to high-frequency noise. Notice the Fourier transform pairs:

$$\Im\left\{\frac{\partial}{\partial x} f(x,y)\right\} = j\, 2\pi\, u\, F(u,v)$$

$$\Im\left\{\nabla^2 f(x,y)\right\} = -\,4\pi^2\,(u^2 + v^2)\, F(u,v)$$

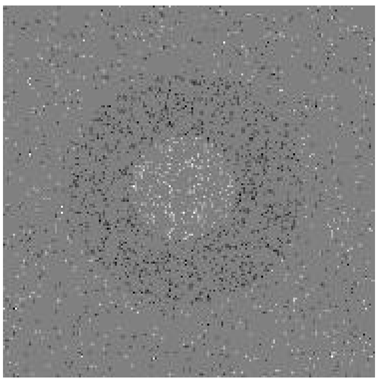
It can be seen that the effect of the first and second order derivatives on the original spectrum is that this will be weighted linearly and quadratic, respectively.





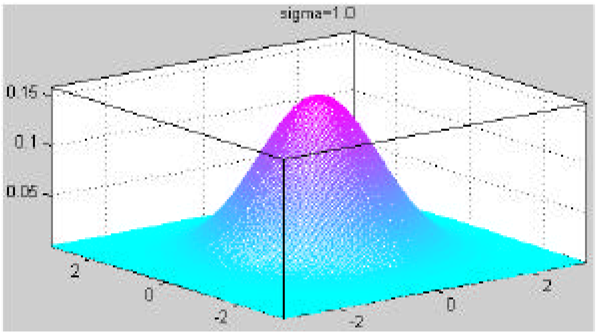
LOG filter (Laplacian of Gaussian)

It has been known since Kuffler (1953) that the spatial organization of the receptive fields of the retina is circularly symmetric with a central excitatory region and an inhibitory surrounding.



Lets try to design a version of the Laplacian which is less sensitive to high-frequency noise. As a starting point, consider the Gaussian function below:

$$G(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



Now compute: $\nabla^2 G(x,y) = \frac{\partial^2}{\partial x^2}[G(x,y)] + \frac{\partial^2}{\partial y^2}[G(x,y)]$

$$A = \frac{x^2+y^2}{2\sigma^2}$$

$$\frac{\partial}{\partial x}[G(x,y)] = \frac{1}{2\pi\sigma^2} \cdot \frac{-2x}{2\sigma^2} \cdot e^{-A}$$

$$= -\frac{x}{2\pi\sigma^4} \cdot e^{-A}$$

$$\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}[G(x,y)]\right\} = \left(-\frac{1}{2\pi\sigma^4} \cdot e^{-A} + \frac{-x}{2\pi\sigma^4} \cdot \frac{-2x}{2\sigma^2}\right) \cdot e^{-A}$$

$$= \left(\frac{x^2}{2\pi\sigma^6} - \frac{1}{2\pi\sigma^4}\right) \cdot e^{-A}$$

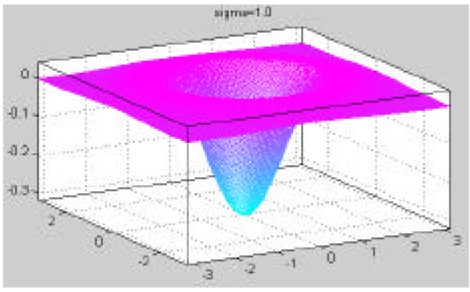
$$\frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}[G(x,y)]\right\} = \left(\frac{y^2}{2\pi\sigma^6} - \frac{1}{2\pi\sigma^4}\right) \cdot e^{-A}$$

$$\nabla^2 G(x,y) = \left(\frac{x^2+y^2}{2\pi\sigma^6} - \frac{1}{\pi\sigma^4}\right) \cdot e^{-\frac{x^2+y^2}{2\sigma^2}}$$

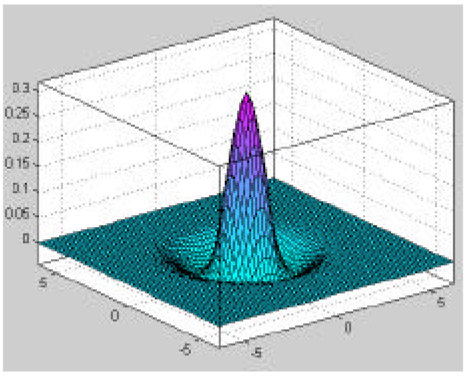
$$r^2 = x^2 + y^2$$

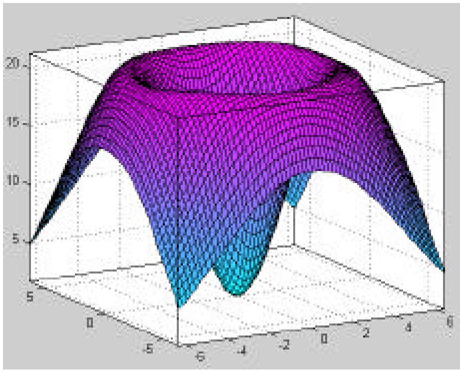
$$\nabla^2 G(x,y) = -\frac{1}{\pi\sigma^4}\left(1 - \frac{r^2}{2\sigma^2}\right) \cdot e^{-\frac{r^2}{2\sigma^2}}$$

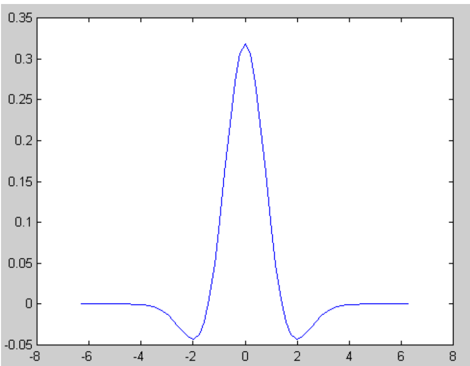
Where $\frac{1}{\pi\sigma^4}$ normalizes the sum of filter coefficients to 1, and σ controls the width of the main lobe.

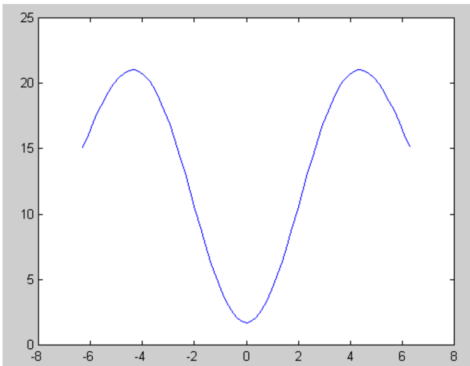


Time domain *Fourier domain*









We see that this function can be used as an edge detector with better properties in the presence of noise. This is due to its band-pass characteristics.

Generally, we have

$$g(x,y) = f(x,y) * h(x,y)$$

Here,

$$h(x,y) = \nabla^2 G(x,y)$$

which gives

$$g(x,y) = \nabla^2 G(x,y) * f(x,y)$$

It can be shown that:

$$\left\{\nabla^2 G(x,y)\right\} * f(x,y) = \nabla^2 \left\{G(x,y) * f(x,y)\right\}$$

This is equivalent to LP-filtering by a Gaussian followed by HP-filtering by a Laplacian.