MOMENTS

Definition

Given a 2D continuous function f(x,y), we define the moment of order (p+q) by the relation:

$$m_{p,q} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q f(x,y)\, dx\, dy \qquad p,q=0,1,2...$$

In the discrete case:

$$m_{p,q} = \sum_x \sum_y x^p y^q f(x,y)$$

Central moments

The central moments are defined as:

$$\mu_{p,q} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{x})^p (y-\bar{y})^q f(x,y)$$

$$\text{where } \bar{x} = \frac{m_{1,0}}{m_{0,0}} \text{ and } \bar{y} = \frac{m_{0,1}}{m_{0,0}}$$

Discrete case: $\mu_{p,q} = \sum_x \sum_y (x-\bar{x})^p (y-\bar{y})^q f(x,y)$

The central moments are invariant under translation

Scale change

Under scale change, *x'*=α *x*, *y'*=α *y*, the moments of f(αx, αy) change to:

$$\mu'_{p,q} = \mu_{p,q} / \alpha^{p+q+2}$$

The normalized moments, defined as:

$$\eta_{p,q} = \frac{\mu'_{p,q}}{\left(\mu'_{0,0}\right)^{\gamma}} \quad \gamma=(p+q+2)/2$$

are then invariant to size change.

Rotation and reflexion

Under a linear coordinate transformation,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

it is possible to find certain polynomials of $\mu_{p,q}$ that remain unchanged under the transformation. For example, some moments invariant with respect to rotation (that is, $\alpha=\delta=\cos\theta$, $\beta=-\gamma=\sin\theta$) and reflexion ($\alpha=-\delta=\cos\theta$, $\beta=\gamma=\sin\theta$) are given as follows.

$$\begin{aligned}
&\text{For } p+q=1: \\
\Phi_0 &= \mu_{0,1} = \mu_{1,0} = 0 \text{ (always invariant)} \\
&\text{For } p+q=2: \\
\Phi_1 &= \mu_{2,0} + \mu_{0,2} \\
\Phi_2 &= \left(\mu_{2,0} - \mu_{0,2}\right)^2 + 4\mu_{1,1}^2 \\
&\text{For } p+q=3: \\
\Phi_3 &= \left(\mu_{3,0} - 3\mu_{1,2}\right)^2 + \left(\mu_{0,3} - 3\mu_{2,1}\right)^2 \\
\Phi_4 &= \left(\mu_{3,0} + \mu_{1,2}\right)^2 + \left(\mu_{0,3} + \mu_{2,1}\right)^2 \\
\Phi_5 &= \left(\mu_{3,0} - 3\mu_{1,2}\right)\left(\mu_{3,0} + \mu_{1,2}\right)\left[\left(\mu_{3,0} + \mu_{1,2}\right)^2 - 3\left(\mu_{0,3} + \mu_{2,1}\right)^2\right] \\
&+ \left(\mu_{0,3} - 3\mu_{2,1}\right)\left(\mu_{0,3} + \mu_{2,1}\right)\left[\left(\mu_{0,3} + \mu_{2,1}\right)^2 - 3\left(\mu_{3,0} + \mu_{1,2}\right)^2\right] \\
\Phi_6 &= \left(\mu_{2,0} - \mu_{0,2}\right)\left[\left(\mu_{3,0} + \mu_{1,2}\right)^2 - \left(\mu_{0,3} + \mu_{2,1}\right)^2\right] + 4\mu_{1,1}\left(\mu_{3,0} + \mu_{1,2}\right)\left(\mu_{0,3} + \mu_{2,1}\right)
\end{aligned}$$