Exercise 1

a)

$$\begin{matrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{matrix} = f_3 = f_1 * f_2$$

b)

$$\Im\left[f_3(x,y)\right] = 4 + 4\cos(u) + 4\cos(v) + 2\cos(u+v) + 2\cos(u-v)$$

c)

$$\begin{aligned}
\Im\left[f_1(x,y)\right] &= 2 + 2\cos(v) \\
\Im\left[f_2(x,y)\right] &= 2 + 2\cos(u) \\
\Im\left[f_3(x,y)\right] &= \Im\left[f_1(x,y)\right] \cdot \Im\left[f_2(x,y)\right] \\
&= (2 + 2\cos(v)) \cdot (2 + 2\cos(u)) \\
&= 4 + 4\cos(u) + 4\cos(v) + 4\cos(u)\cos(v) \\
&= 4 + 4\cos(u) + 4\cos(v) + 2\cos(u+v) + 2\cos(u-v)
\end{aligned}$$

d)

The Fourier transform of the median filter does not exist. This is because it is a non-linear filter.

Exercise 2

a)
Refer to the page 'The Laplace-operator in the frequency domain' where the general solution is derived from a nxn filter of any type (LP, HP, BP).

b)

1	2	1
2	5	2
1	2	1

c)

0	1	0
1	1	1
0	1	0

d)
The median may be superior to the mean if the image has been subject to salt & pepper noise.

Exercise 3

a)

$$\begin{array}{ccc} & h2 & \\ 0 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 0 \end{array}$$

b)

$$\begin{array}{ccccc} & & f/2 & & \\ & d/2 & b/2 & c/2 & \\ e/2 & a/2 & g & a/2 & e/2 \\ & c/2 & b/2 & d/2 & \\ & & f/2 & & \end{array}$$

c)

$$\begin{array}{ccc} 0 & 2 & 0 \\ 2 & 5 & 3 \\ 0 & 5 & 0 \end{array}$$

Exercise 4

a)

$$
\begin{array}{ccc} & h2 & \\ 0 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 0 \end{array}
\quad \longrightarrow \quad
\begin{array}{ccccc} & & h3 & & \\ & 1 & 2 & 1 & \\ 1 & 8 & 14 & 8 & 1 \\ 2 & 14 & 24 & 14 & 2 \\ 1 & 8 & 14 & 8 & 1 \\ & 1 & 2 & 1 & \end{array}
$$

b)

$$
\begin{aligned}
& 24 \\
& + 28 \cos(u) + 28 \cos(v) \\
& + 4 \cos(2u) + 4 \cos(2v) \\
& +16 \cos(u+v) + 16 \cos(u-v) \\
& + 2 \cos(2u+v) + 2 \cos(2u-v) \\
& + 2 \cos(u+2v) + 2 \cos(u-2v)
\end{aligned}
$$

Exercise 5

a)

Vertical 2:	-2
Horizontal 2:	-2
Diagonal 2:	-4
Single 2:	-8

b)

The operator is obviously not rotation-invariant.

c)

$\Im(u,v)\left[h(x,y)\right]$ = - 4 + 2 (cos(u) + cos(v))

d)

$$\Im\left[h(x,y)\right]_{(u=\pi,v=0)} = 2(-2+\cos\pi+\cos 0) = -4$$

$$\Im\left[h(x,y)\right]_{(u=\pi,v=0)} = 2(-2+\cos\pi+\cos\pi) = -8$$

e)

Again, it can be seen that the operator is not rotational invariant.

Exercise 6

a)

h_5, which is a low-pass filter, can be used to eliminate the noise.
h_2, which is a high-pass filter (Laplacian-type), can be used to create the gradient image.

b)

If we denote f the input image, and g the output image, we can write:

$$g = (f * h_1) * h_2 = f * (h_1 * h_2) = f * h_{new}$$

c)

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & 0 & 2 & \\ 1 & 0 & -12 & 0 & 1 \\ & 2 & 0 & 2 & \\ & & 1 & & \end{array}$$

d)

The new filter has a band-pass type.

e)

An FFT for an image of NxN pixels requires $\mathbf{2N^2\log_2N}$ real multiplications.
In total (FFT multiplication with complex numbers => inverse FFT), we need: $\mathbf{2N^2\log_2N+4N^2+2N^2\log_2N=4N^2(1+\log_2N)}$ operations. Convolution, or filtering in the spatial domain (n coeff <>0), takes $\mathbf{N^2n}$ multiplications.
Therefore, filtering in the spatial domain is computationally more efficient for cases where $\mathbf{n< 4(1+\log_2N)}$, for example n<40 for N=512 means a maximal sized operator of 6 x 6 can be used.

Exercise 7

a) $F(u,v)=(2+\cos(u)+\cos(v))/4$

b)

1	2	1
2	5	2
1	2	1

c)

0	1	0
1	1	1
0	1	0

d)
The median filter can remove ’spikes’, or salt-and-pepper noise, but also removes corners of an object.

Exercise 8

a)

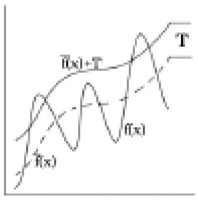
Denote the output image by g(i,j), then:

$$g(i,j) = -4f(i,j) + f(i+1,j) + f(i-1,j) + f(i,j) + f(i,j+1) + f(i,j-1)$$

$$g(i,j) = -5\left[f(i,j) - \tfrac{1}{5}\left[f(i+1,j) + f(i-1,j) + f(i,j) + f(i,j+1) + f(i,j-1) \right] \right]$$

$$g(i,j) = -5\left[f(i,j) - \bar{f}(i,j) \right]$$

b)



$$g(x) = 1 \quad \text{if} \quad f(x) \geq \bar{f}(x) + T$$

Equivalent to

$$\underbrace{f(x) - \bar{f}(x)}_{\text{Laplace}} \geq +T$$

c) Assume that g_k is degraded by the effect of unfocused section at levels k+1 and k-1 so that

$$g_k = f_k + \varepsilon(\bar{f}_{k+1} + \bar{f}_{k-1})$$

Then, a reasonable estimation of f_k would be:

$$\hat{f}_k = g_k - \varepsilon(\bar{g}_{k+1} + \bar{g}_{k-1})$$

Exercise 9

a)
Let f be the input image, g_1 the output after the first convolution, and g_2 the output after the second convolution. Then,

$$
\begin{aligned}
g_1(x,y) &= f(x,y) * h_1(x,y) \\
g_2(x,y) &= g_1(x,y) * h_2(x,y) \\
&= \{f(x,y) * h_1(x,y)\} * h_2(x,y) \\
&= f(x,y) * \{h_1(x,y) * h_2(x,y)\} \\
&= f(x,y) * h_3(x,y)
\end{aligned}
$$

b)

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 2 & 8 & 2 & \\
1 & 8 & 20 & 8 & 1 \\
 & 2 & 8 & 2 & \\
 & & 1 & &
\end{array}
$$

c) The filter should be normalized so that the image is not 'amplified' (the intensity is not increased globally). The normalization factor k will be equal to the sum of all filter coefficients. In the present case, k=8

d) For example, $\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{array}$

as this filter removes noise along the vertical edges without blurring across