Solution

Let ε denote the error between data f(x,y) and model g(x,y). By the least sqaure solution, we would like to minimize the squared error:

$$\varepsilon^2 = \sum [f(x,y) - g(x,y)]^2 = \sum [f(x,y) - a_0 - a_1 x - a_2 y]^2$$

We determine the parameters a_0, a_1 and a_2 by solving the following equations:

$$\frac{\partial \varepsilon^2}{\partial a_0} = 0 \quad \frac{\partial \varepsilon^2}{\partial a_1} = 0 \quad \frac{\partial \varepsilon^2}{\partial a_2} = 0$$

Solution

$$\frac{\partial \varepsilon^2}{\partial a_0} = 2 \cdot \sum \left[f(x,y) - a_0 - a_1 x - a_2 y \right]$$

$$= 2 \cdot \sum f(x,y) - 2 \underbrace{\sum a_0}_{Na_0} - 2a_1 \underbrace{\sum x}_{0} - 2a_2 \underbrace{\sum y}_{0}$$

$$a0 = \frac{1}{N} \sum f(x,y) \text{ (mean intensity)}$$

(N is the number of pixels)

Solution

$$\frac{\partial \varepsilon^2}{\partial a_1} = 2 \cdot \sum \left[xf(x,y) - a_0 x - a_1 x^2 - a_2 xy \right]$$

$$= 2 \cdot \sum xf(x,y) - 2a_1 \sum x^2 - 2a_0 \underbrace{\sum x}_{0} - 2a_2 \underbrace{\sum xy}_{0}$$

$$a_1 = \frac{\sum x\, f(x,y)}{\sum x^2} \text{ and similarly } a_2 = \frac{\sum y\, f(x,y)}{\sum y^2}$$

Solution

Notice that a_1 and a_2 can be looked upon as f_x’ and f_y’, respectively, in the image region:

$$f_x^{'}: \begin{matrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{matrix} \; /6 \qquad f_y^{'}: \begin{matrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{matrix} \; /6$$

Solution

The slope of the plane then equals the gradient magnitude:

$$\left|G_f\right| = \sqrt{f_x^{'2} + f_y^{'2}}$$

The orientation equals the gradient orientation:

$$\theta = \arctan\left(f_y^{'} / f_x^{'}\right) = \angle G_f$$

Solution

Numerically, we obtain f_x' and f_y' by convolving the image region with the operators above:

$$f_x^{'} = (-17 - 15 - 10 + 40 + 30 + 26) / 26 = 54 / 6$$

$$f_y^{'} = (-10 - 16 - 26 + 17 + 23 + 40) / 6 = 28 / 6$$

$$\left| G_f \right| \simeq 9.3 \text{ (grayscale units per length unit)}$$

$$\theta \simeq 30^{o}$$