Point Distribution Model

Solution

A. The PDM (Point Distribution Model)

The mean shape

$$\bar{\mathbf{x}} = \frac{1}{5}\sum_{i=1}^{5}\mathbf{x}_i = \frac{1}{5}\left(\begin{bmatrix}4\\2\end{bmatrix}+\begin{bmatrix}3\\3\end{bmatrix}+\begin{bmatrix}2\\3\end{bmatrix}+\begin{bmatrix}1\\5\end{bmatrix}+\begin{bmatrix}1\\4\end{bmatrix}\right)=\begin{bmatrix}2.2\\3.4\end{bmatrix}$$

The covariance matrix

$$\mathbf{C} = \frac{1}{5-1}\sum_{i=1}^{5}(\mathbf{x}_i-\bar{\mathbf{x}})^T(\mathbf{x}_i-\bar{\mathbf{x}}) =$$

$$\frac{1}{4}\left(\left(\begin{bmatrix}4\\2\end{bmatrix}-\begin{bmatrix}2.2\\3.4\end{bmatrix}\right)\left(\begin{bmatrix}4\\2\end{bmatrix}-\begin{bmatrix}2.2\\3.4\end{bmatrix}\right)^T+\ldots+\left(\begin{bmatrix}1\\4\end{bmatrix}-\begin{bmatrix}2.2\\3.4\end{bmatrix}\right)\left(\begin{bmatrix}1\\4\end{bmatrix}-\begin{bmatrix}2.2\\3.4\end{bmatrix}\right)^T\right)=\begin{bmatrix}1.7 & -1.35\\-1.35 & 1.3\end{bmatrix}$$

Eigenvalue Decomposition

Eigenvalues:

$$|\mathbf{C}-l\mathbf{I}| = \begin{vmatrix}\mathbf{C}_{11}-l & \mathbf{C}_{12}\\ \mathbf{C}_{21} & \mathbf{C}_{22}-l\end{vmatrix} = (\mathbf{C}_{11}-l)(\mathbf{C}_{22}-l)-(\mathbf{C}_{12})^2=0$$

$$\Rightarrow \underbrace{1}_{a}\, l^2 + \underbrace{(-\mathbf{C}_{11}-\mathbf{C}_{22})}_{b}\, l + \underbrace{(\mathbf{C}_{11}\mathbf{C}_{22}-\mathbf{C}_{12}{}^2)}_{c} = 0$$

$$l_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a} \Rightarrow \begin{cases} l_1 = 2.8647 \\ l_2 = 0.135 \end{cases}$$

Check: 2.8647 + 0.1353 = 1.7 + 1.3 = 3.

Eigenvectors:

$$\mathbf{C}\mathbf{p}_1 = l_1\mathbf{p}_1 \Rightarrow \begin{bmatrix}\mathbf{C}_{11} & \mathbf{C}_{12}\\ \mathbf{C}_{21} & \mathbf{C}_{22}\end{bmatrix}\begin{bmatrix}p_{11}\\p_{12}\end{bmatrix} = l_1\begin{bmatrix}p_{11}\\p_{12}\end{bmatrix} \xrightarrow{p_{11}=1} p_{12} = \frac{l_1-\mathbf{C}_{11}}{\mathbf{C}_{12}} = \text{-}0.8628$$

$$\mathbf{p}_1 = \begin{bmatrix}1\\ \text{-}0.8628\end{bmatrix} \overset{normalize}{\Rightarrow} \mathbf{p}_1 = \frac{\begin{bmatrix}1\\ \mathrm{p}_{12}\end{bmatrix}}{\sqrt{1+\mathrm{p}_{12}^2}} = \begin{bmatrix}0.7571\\ \text{-}0.6532\end{bmatrix}$$

$$\mathbf{C}\mathbf{p}_2 = l_2\mathbf{p}_2 \Rightarrow \begin{bmatrix}\mathbf{C}_{11} & \mathbf{C}_{12}\\ \mathbf{C}_{21} & \mathbf{C}_{22}\end{bmatrix}\begin{bmatrix}p_{21}\\p_{22}\end{bmatrix} = l_2\begin{bmatrix}p_{21}\\p_{22}\end{bmatrix} \xrightarrow{p_{21}=1} p_{22} = \frac{l_2-\mathbf{C}_{11}}{\mathbf{C}_{12}} = 1.1591$$

$$\begin{bmatrix}1\\ \ldots\end{bmatrix} \overset{normalize}{\Rightarrow} \frac{\begin{bmatrix}1\\ \mathrm{p}_{22}\end{bmatrix}}{\ldots} \quad \begin{bmatrix}0.6532\\ \ldots\end{bmatrix}$$

C. PDM Reduction and Approximating a New Shape

Using the first variation mode (explaining ~95% of the total variation) we get the reduced PDM:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \bar{\mathbf{x}} + b_1\mathbf{p}_1 = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} + b_1\begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix} = \begin{bmatrix} \bar{x} + b_1 p_{11} \\ \bar{y} + b_1 p_{12} \end{bmatrix}$$

Minimizing the distance between the reduced model and the new shape with respect to b_1:

$$D = ((\bar{x} + p_{11}b_1) - x_{new})^2 + ((\bar{y} + p_{12}b_1) - y_{new})^2$$

$$\frac{\partial D}{\partial b_1} = 2\bar{x}p_{11} + 2b_1 p_{11}^2 - 2x_{new} p_{11} + 2\bar{y}p_{12} + 2b_1 p_{12}^2 - 2y_{new} p_{12} = 0$$

$$b_1 = \frac{-\bar{x}p_{11} + x_{new} p_{11} - \bar{y}p_{12} + y_{new} p_{12}}{p_{11}^2 + p_{12}^2} = 0.7631$$

or

$$\mathbf{x}_{new} \approx \bar{\mathbf{x}} + b_1\mathbf{p}_1 \Rightarrow b_1 = \mathbf{p}_1^T(\mathbf{x}_{new} - \bar{\mathbf{x}}) = 0.7631$$

The estimate of the new shape:

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} \bar{x} + p_{11}b_1 \\ \bar{y} + p_{12}b_1 \end{bmatrix} = \begin{bmatrix} 2.7778 \\ 2.9015 \end{bmatrix}$$