LAPLACE OPERATOR

Laplace operator (spatial domain)

The Laplace operator is defined by: $\nabla^2 f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

In the discrete case: $\nabla^2 f(i,j) \equiv \Delta_x^2 f(i,j) + \Delta_y^2 f(i,j)$

where

$$
\begin{aligned}
\Delta_x f(i,j) &\equiv f(i,j) - f(i-1,j) \\
\Delta_y f(i,j) &\equiv f(i,j) - f(i,j-1) \\
\Delta_x^2 f(i,j) &\equiv \Delta_x f(i+1,j) - \Delta_x f(i,j) \\
&\equiv \left[f(i+1,j) - f(i,j)\right] - \left[f(i,j) - f(i-1,j)\right] \\
&\equiv f(i+1,j) + f(i-1,j) - 2f(i,j) \\
\Delta_y^2 f(i,j) &\equiv f(i,j+1) + f(i,j-1) - 2f(i,j)
\end{aligned}
$$

Laplace operator

It follows that:

$$\nabla^2 f \equiv \left[f(i+1,j) + f(i-1,j) + f(i,j+1) + f(i,j-1)\right] - 4f(i,j)$$

Notice that this result is proportional to:

$$f(i,j) - \tfrac{1}{5}\left[f(i+1,j) + f(i-1,j) + f(i,j) + f(i,j+1) + f(i,j-1)\right]$$

Hence, the discrete Laplace operator can be replaced by the original function subtracted by an average of this function in a small neighborhood:

$$\nabla^2 f = f(i,j) - \bar{f}(i,j)$$

Laplace operator (freq. domain)

$$f(X_1, X_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} U_0(X_1 - m\Delta) U_0(X_2 - n\Delta)$$

where $U_0 = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$ (Kronecker delta)

$$F(jU_1, jU_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} U_0(X_1 - m\Delta) U_0(X_2 - n\Delta) e^{-j(U_1 X_1 + U_2 X_2)} dX_1 dX_2$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} \int_{-\infty}^{\infty} U_0(X_1 - m\Delta) e^{-jU_1 X_1} dX_1 \int_{-\infty}^{\infty} U_0(X_2 - n\Delta) e^{-jU_2 X_2} dX_2$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} e^{-j(U_1 m\Delta + U_2 n\Delta)}$$

$$\left(\int_{-\infty}^{\infty} f(X) U_0(X - X') dX = f(X') \right)$$

Laplace operator

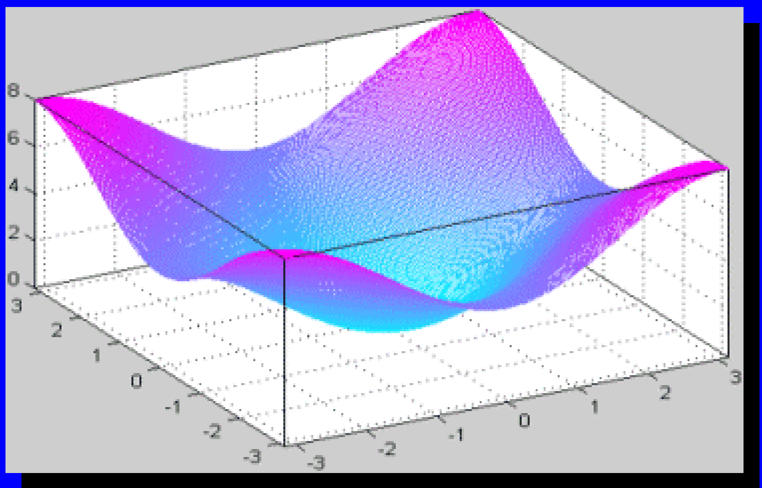
The Laplace operator is defined by:

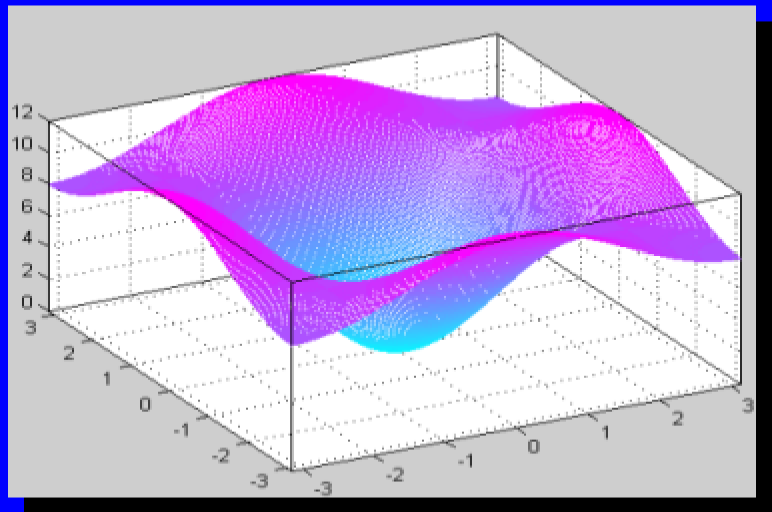
$$
\begin{array}{lll}
a_{-1,1} = 0 & a_{0,1} = -1 & a_{1,1} = 0 \\
a_{-1,0} = -1 & a_{0,0} = 4 & a_{1,0} = -1 \\
a_{-1,-1} = 0 & a_{0,-1} = -1 & a_{1,-1} = 0
\end{array}
$$

$$
\begin{aligned}
F(jU_1, jU_2) &= -e^{j\Delta U_2} - e^{j\Delta U_1} + 4 - e^{-j\Delta U_2} - e^{-j\Delta U_1} \\
&= 4 - 2\cos \Delta U_1 - 2\cos \Delta U_2 \\
&= 2(2 - \cos \Delta U_1 - \cos \Delta U_2)
\end{aligned}
$$

Laplace operator

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$





Laplace operator

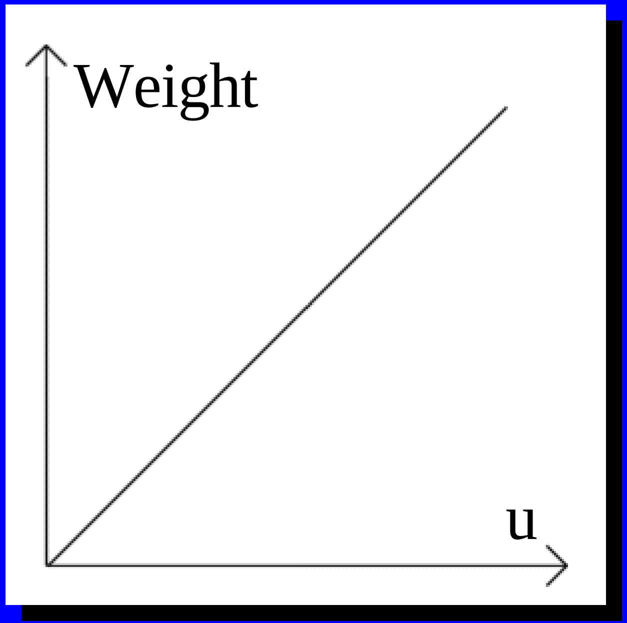
- Sensitive to high-frequency noise.
- Notice the Fourier transform pairs:

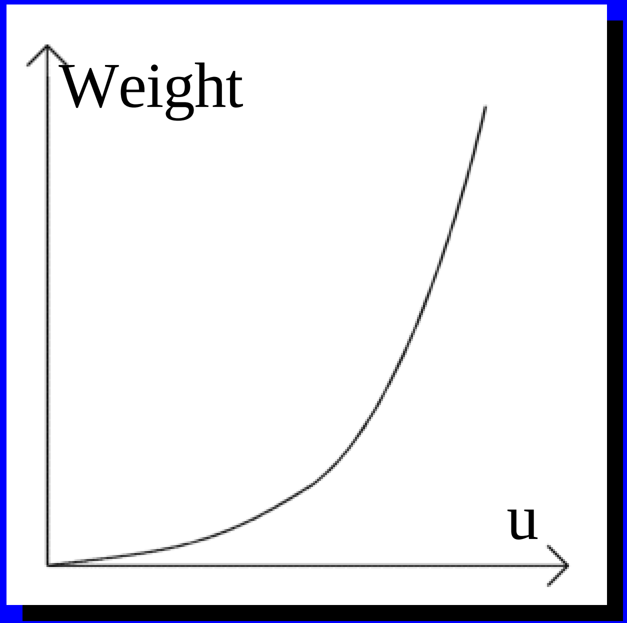
$$\Im\left\{\frac{\partial}{\partial x} f(x,y)\right\} = j\,2\pi\,u\,F(u,v)$$

$$\Im\left\{\nabla^2 f(x,y)\right\} = -\,4\pi^2\,(u^2 + v^2)\,F(u,v)$$

Laplace operator

It can be seen that the effect of the first and second order derivatives on the original spectrum is that this will be weighted linearly and quadratic, respectively:

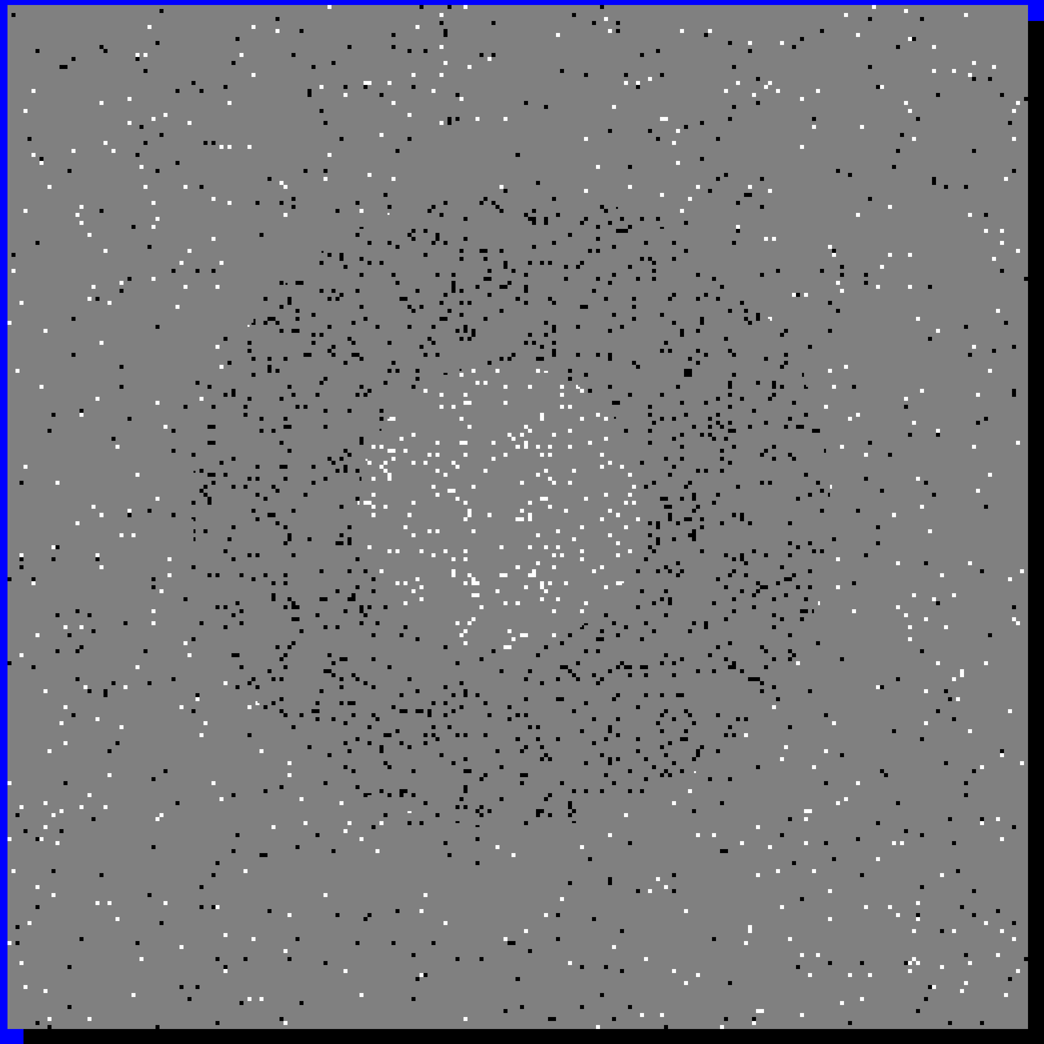




LOG filter (Laplacian of Gaussian)

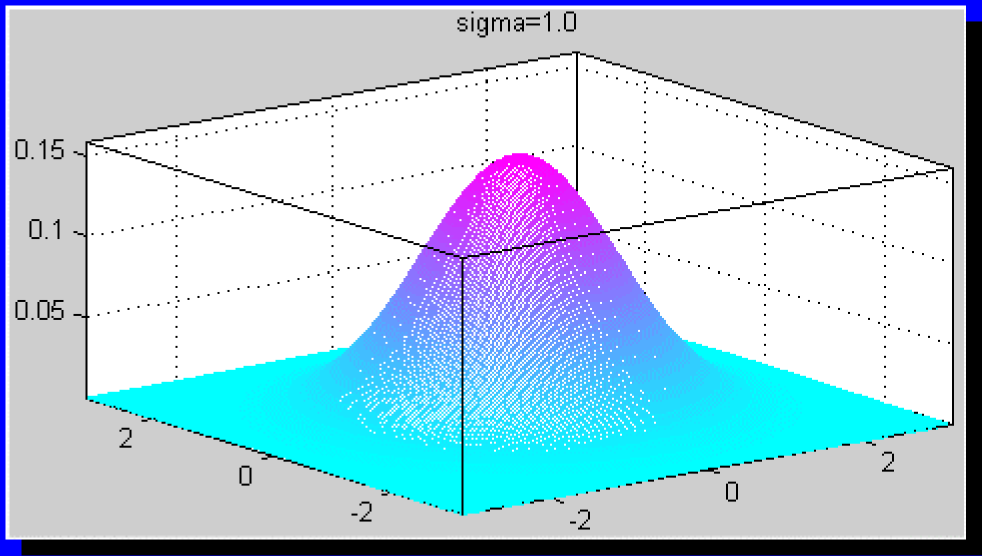
- It has been known since Kuffler (1953) that the spatial organization of the receptive fields of the retina is circulary symmetric with a central excitatory region and an inhibitory surround.

LOG filter (Laplacian of Gaussian)



LOG filter (Laplacian of Gaussian)

$$G(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



Compute: $$\nabla^2 G(x,y) = \frac{\partial^2}{\partial x^2}[G(x,y)] + \frac{\partial^2}{\partial y^2}[G(x,y)]$$

LOG filter (Laplacian of Gaussian)

$$\frac{\partial}{\partial x}[G(x,y)] = \frac{1}{2\pi\sigma^2}\cdot\frac{-2x}{2\sigma^2}\cdot e^{-A} \qquad A = \frac{x^2+y^2}{2\sigma^2}$$

$$= -\frac{x}{2\pi\sigma^4}\cdot e^{-A}$$

$$\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}[G(x,y)]\right\} = \left(-\frac{1}{2\pi\sigma^4}\cdot e^{-A} + \frac{-x}{2\pi\sigma^4}\cdot\frac{-2x}{2\sigma^2}\right)\cdot e^{-A}$$

$$= \left(\frac{x^2}{2\pi\sigma^6} - \frac{1}{2\pi\sigma^4}\right)\cdot e^{-A}$$

$$\frac{\partial}{\partial y}\left\{\frac{\partial}{\partial y}[G(x,y)]\right\} = \left(\frac{y^2}{2\pi\sigma^6} - \frac{1}{2\pi\sigma^4}\right)\cdot e^{-A}$$

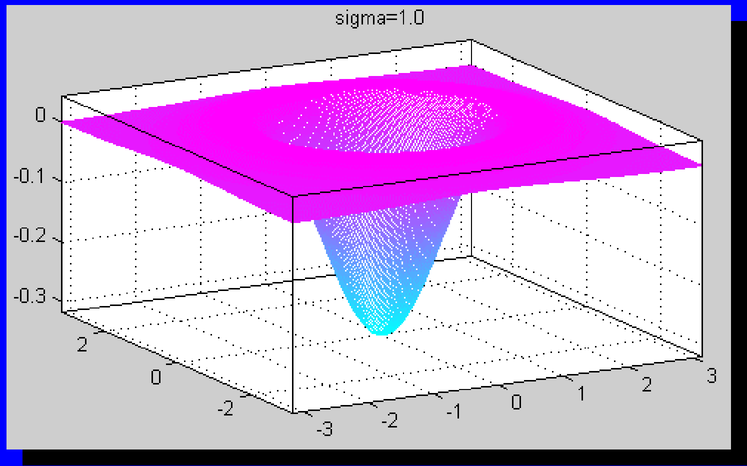
$$\nabla^2 G(x,y) = \left(\frac{x^2+y^2}{2\pi\sigma^6} - \frac{1}{\pi\sigma^4}\right)\cdot e^{-\frac{x^2+y^2}{2\sigma^2}}$$

LOG filter (Laplacian of Gaussian)

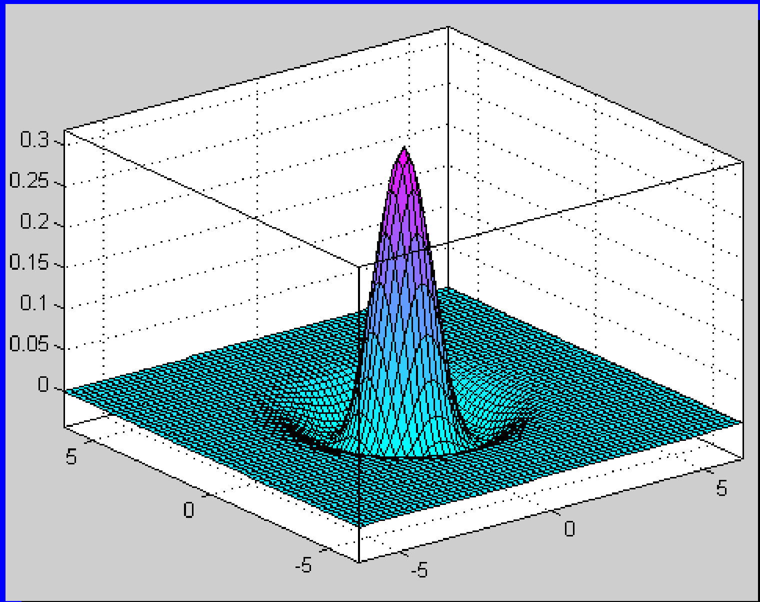
$$r^2 = x^2 + y^2$$

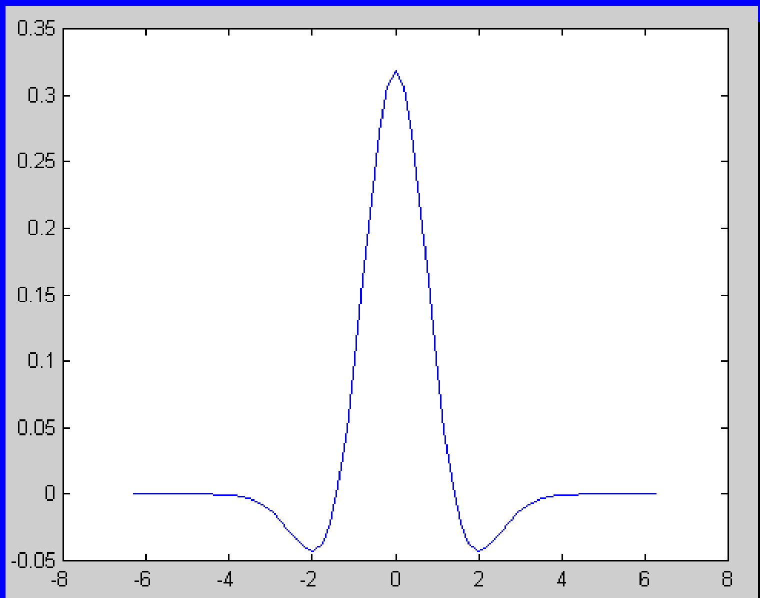
$$\nabla^2 G(x,y) = -\frac{1}{\pi\sigma^4}\left(1 - \frac{r^2}{2\sigma^2}\right) \cdot e^{-\frac{r^2}{2\sigma^2}}$$

Where $\frac{1}{\pi\sigma^4}$ normalizes the sum of filter coefficients to 1, and σ controls the width of the main lobe.

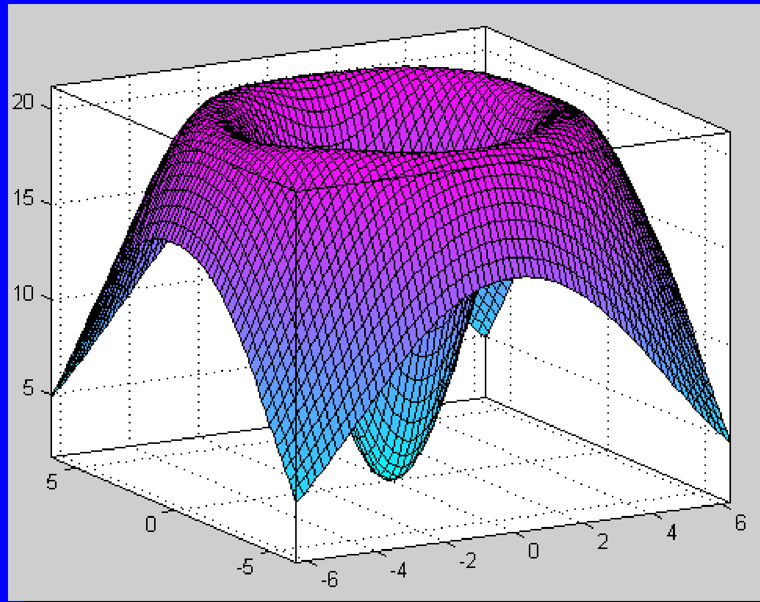


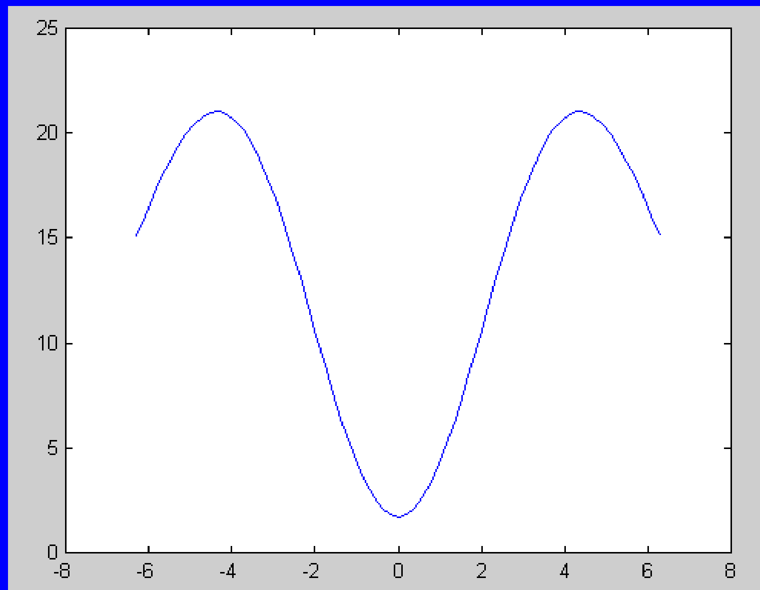
Time domain





Fourier domain





LOG filter (Laplacian of Gaussian)

Generally, $g(x,y) = f(x,y) * h(x,y)$

Here, $h(x,y) = \nabla^2 G(x,y)$

which gives $g(x,y) = \nabla^2 G(x,y) * f(x,y)$

It can be shown that:

$$\left\{\nabla^2 G(x,y)\right\} * f(x,y) = \nabla^2 \left\{G(x,y) * f(x,y)\right\}$$

This is equivalent to LP-filtering by Gaussian followed by HP-filtering by Laplacian.