

LAPLACE OPERATOR

Laplace operator (spatial domain)

The Laplace operator is defined by: $\nabla^2 f \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$

In the discrete case: $\nabla^2 f(i, j) \equiv \Delta_x^2 f(i, j) + \Delta_y^2 f(i, j)$

where $\Delta_x f(i, j) \equiv f(i, j) - f(i-1, j)$

$$\Delta_y f(i, j) \equiv f(i, j) - f(i, j-1)$$

$$\Delta_x^2 f(i, j) \equiv \Delta_x f(i+1, j) - \Delta_x f(i, j)$$

$$\equiv [f(i+1, j) - f(i, j)] - [f(i, j) - f(i-1, j)]$$

$$\equiv f(i+1, j) + f(i-1, j) - 2f(i, j)$$

$$\Delta_y^2 f(i, j) \equiv f(i, j+1) + f(i, j-1) - 2f(i, j)$$

Laplace operator

It follows that:

$$\nabla^2 f \equiv [f(i+1, j) + f(i-1, j) + f(i, j+1) + f(i, j-1)] - 4f(i, j)$$

Notice that this result is proportional to:

$$f(i, j) - \frac{1}{5} [f(i+1, j) + f(i-1, j) + f(i, j) + f(i, j+1) + f(i, j-1)]$$

Hence, the discrete Laplace operator can be replaced by the original function subtracted by an average of this function in a small neighborhood:

$$\nabla^2 f = f(i, j) - \bar{f}(i, j)$$

Laplace operator (freq. domain)

$$f(X_1, X_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} U_0(X_1 - m\Delta) U_0(X_2 - n\Delta)$$

where $U_0 = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$ (Kronecker delta)

$$\begin{aligned} F(jU_1, jU_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} U_0(X_1 - m\Delta) U_0(X_2 - n\Delta) e^{-j(U_1 X_1 + U_2 X_2)} dX_1 dX_2 \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} \int_{-\infty}^{\infty} U_0(X_1 - m\Delta) e^{-jU_1 X_1} dX_1 \int_{-\infty}^{\infty} U_0(X_2 - n\Delta) e^{-jU_2 X_2} dX_2 \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} e^{-j(U_1 m\Delta + U_2 n\Delta)} \quad \left(\int_{-\infty}^{\infty} f(X) U_0(X - X') dX = f(X') \right) \end{aligned}$$

Laplace operator

The Laplace operator is defined by:

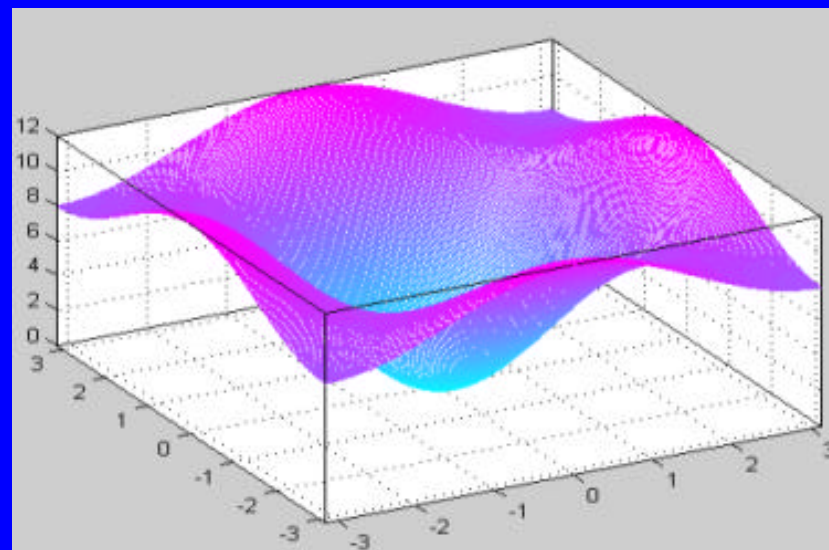
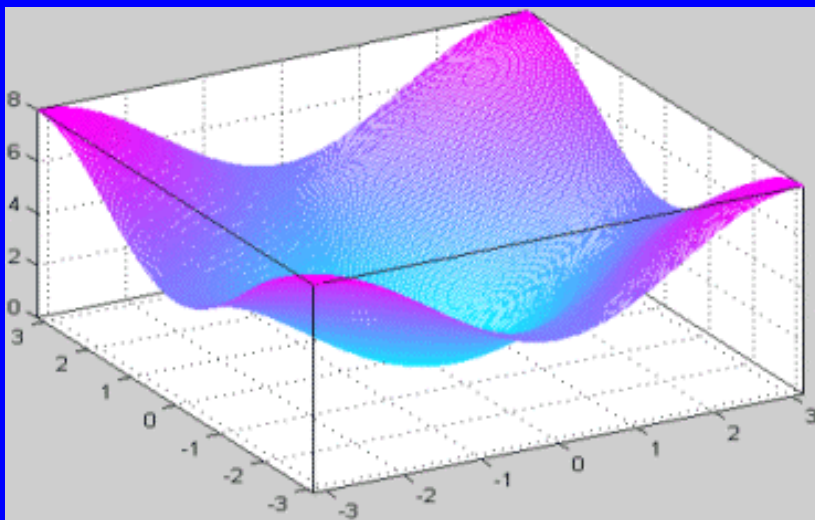
$a_{-1,1} = 0$	$a_{0,1} = -1$	$a_{1,1} = 0$
$a_{-1,0} = -1$	$a_{0,0} = 4$	$a_{1,0} = -1$
$a_{-1,-1} = 0$	$a_{0,-1} = -1$	$a_{1,-1} = 0$

$$\begin{aligned} F(jU_1, jU_2) &= -e^{j\Delta U_2} - e^{j\Delta U_1} + 4 - e^{-j\Delta U_2} - e^{-j\Delta U_1} \\ &= 4 - 2\cos \Delta U_1 - 2\cos \Delta U_2 \\ &= 2(2 - \cos \Delta U_1 - \cos \Delta U_2) \end{aligned}$$

Laplace operator

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$



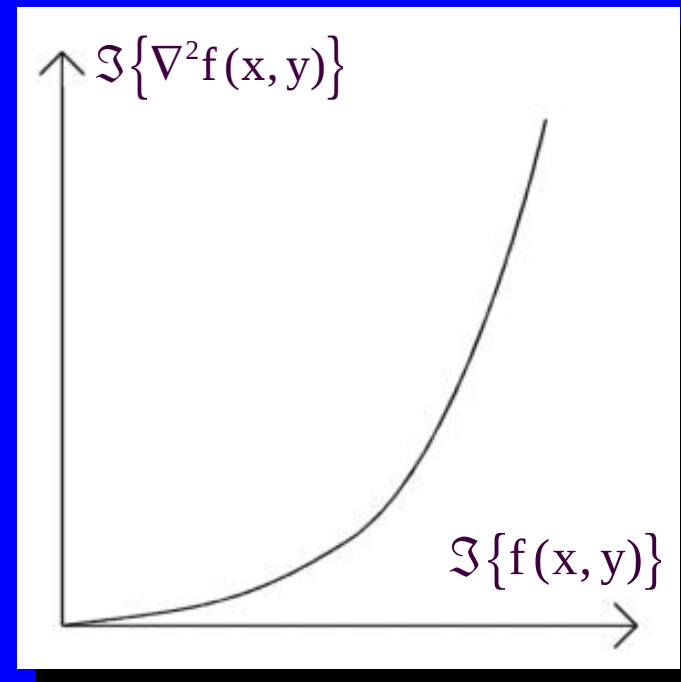
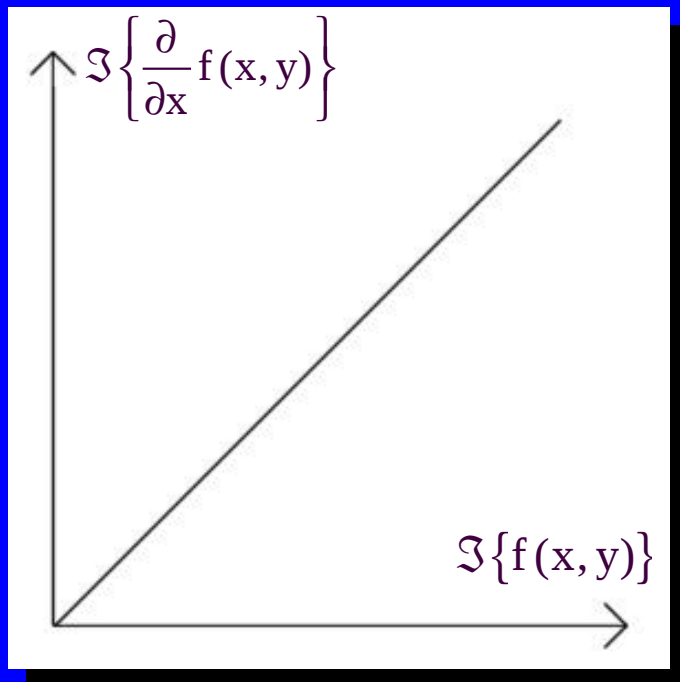
Laplace operator

- Sensitive to high-frequency noise.
- Notice the Fourier transform pairs:

$$\mathfrak{F}\left\{\frac{\partial}{\partial x}f(x,y)\right\} = j 2\pi u F(u,v)$$

$$\mathfrak{F}\{\nabla^2 f(x,y)\} = -4\pi^2 (u^2 + v^2) F(u,v)$$

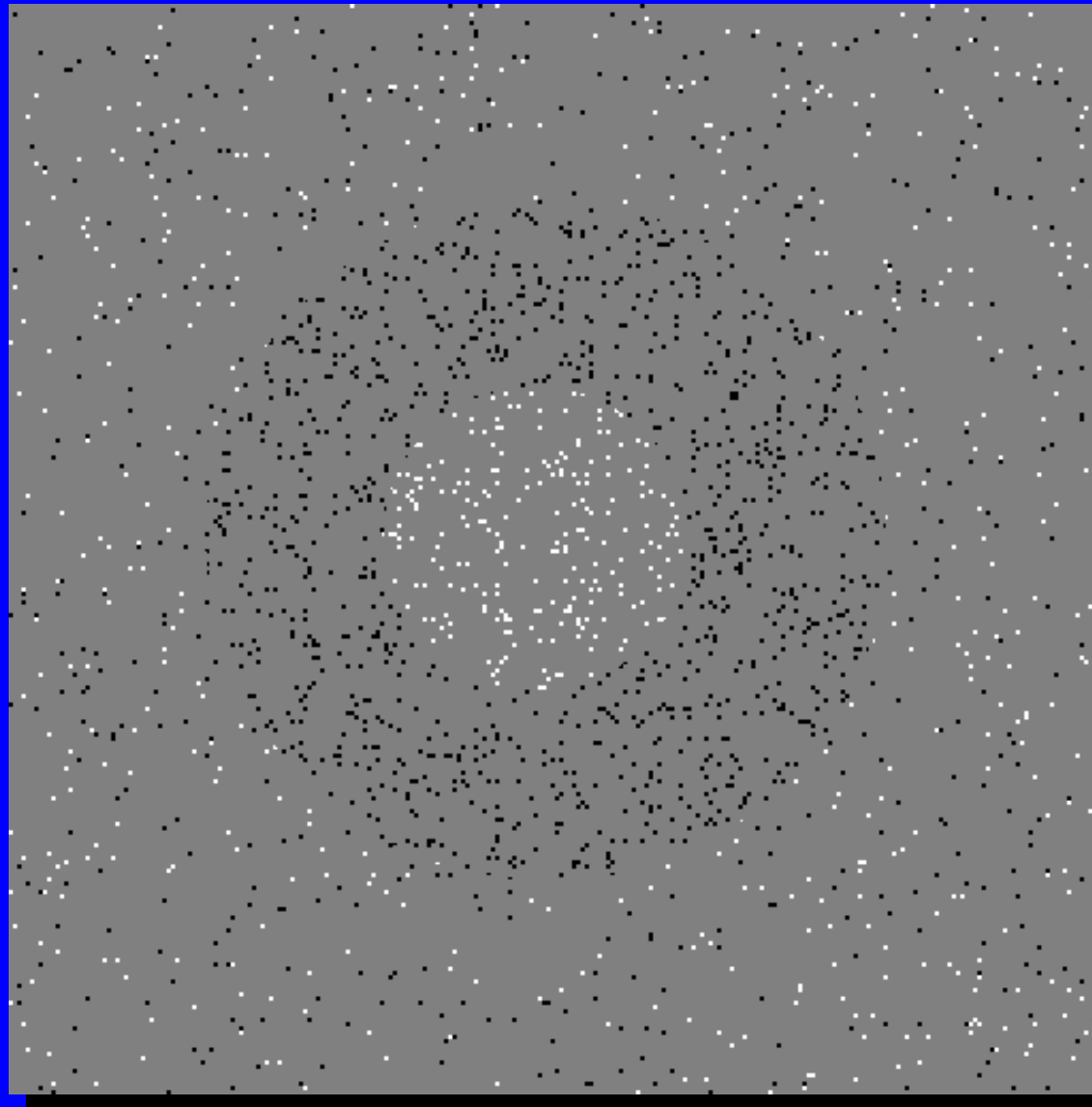
Laplace operator



LOG filter (Laplacian of Gaussian)

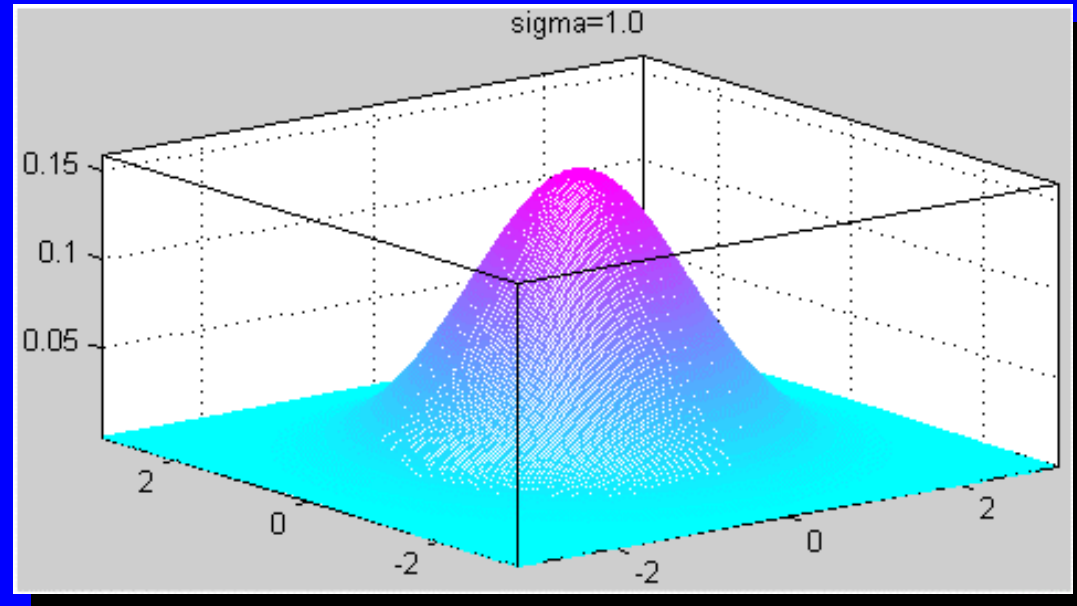
- It has been known since Kuffler (1953) that the spatial organization of the receptive fields of the retina is circularly symmetric with a central excitatory region and an inhibitory surround.

LOG filter (Laplacian of Gaussian)



LOG filter (Laplacian of Gaussian)

$$G(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



Compute: $\nabla^2 G(x, y) = \frac{\partial^2}{\partial x^2} [G(x, y)] + \frac{\partial^2}{\partial y^2} [G(x, y)]$

LOG filter (Laplacian of Gaussian)

$$\frac{\partial}{\partial x}[G(x, y)] = \frac{1}{2\pi\sigma^2} \cdot \frac{-2x}{2\sigma^2} \cdot e^{-A} \quad A = \frac{x^2 + y^2}{2\sigma^2}$$

$$= -\frac{x}{2\pi\sigma^4} \cdot e^{-A}$$

$$\frac{\partial}{\partial x} \left\{ \frac{\partial}{\partial x} [G(x, y)] \right\} = \left(-\frac{1}{2\pi\sigma^4} \cdot e^{-A} + \frac{-x}{2\pi\sigma^4} \cdot \frac{-2x}{2\sigma^2} \right) \cdot e^{-A}$$

$$= \left(\frac{x^2}{2\pi\sigma^6} - \frac{1}{2\pi\sigma^4} \right) \cdot e^{-A}$$

$$\frac{\partial}{\partial y} \left\{ \frac{\partial}{\partial y} [G(x, y)] \right\} = \left(\frac{y^2}{2\pi\sigma^6} - \frac{1}{2\pi\sigma^4} \right) \cdot e^{-A}$$

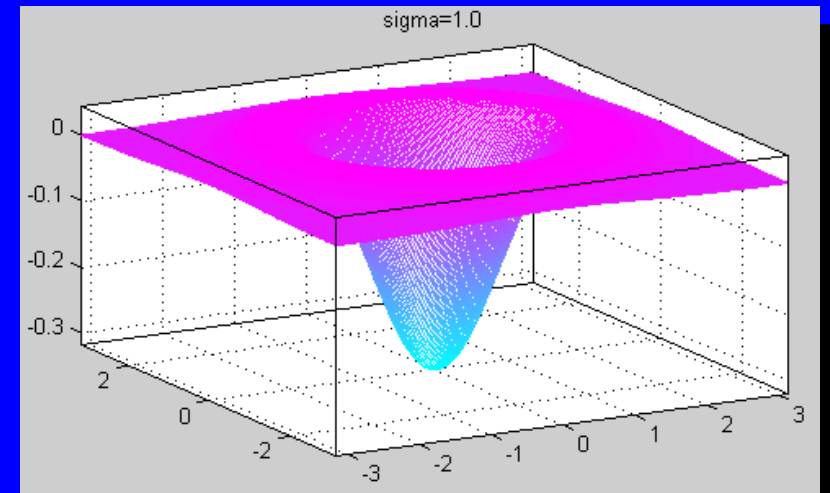
$$\nabla^2 G(x, y) = \left(\frac{x^2 + y^2}{2\pi\sigma^6} - \frac{1}{\pi\sigma^4} \right) \cdot e^{-\frac{x^2 + y^2}{2\sigma^2}}$$

LOG filter (Laplacian of Gaussian)

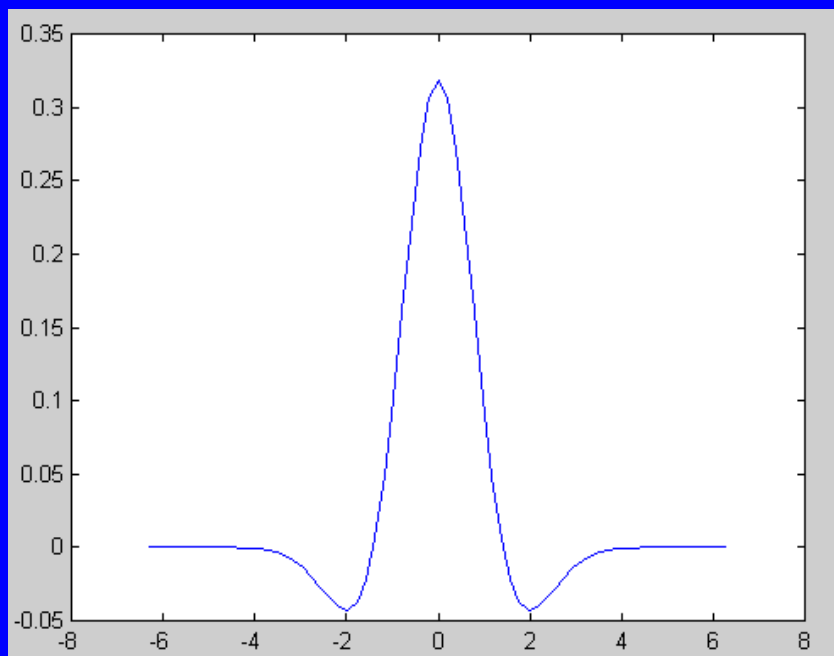
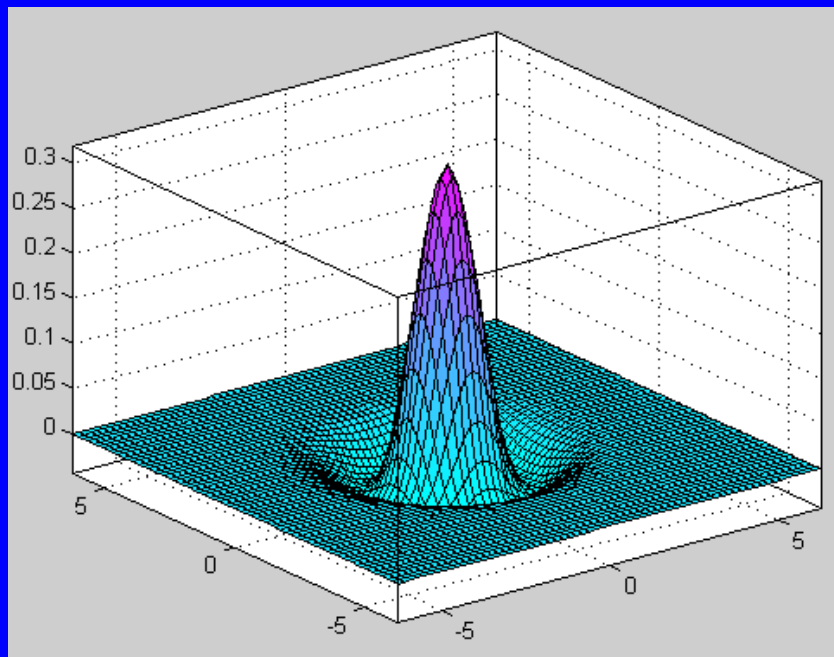
$$r^2 = x^2 + y^2$$

$$\nabla^2 G(x, y) = -\frac{1}{\pi\sigma^4} \left(1 - \frac{r^2}{2\sigma^2} \right) \cdot e^{-\frac{r^2}{2\sigma^2}}$$

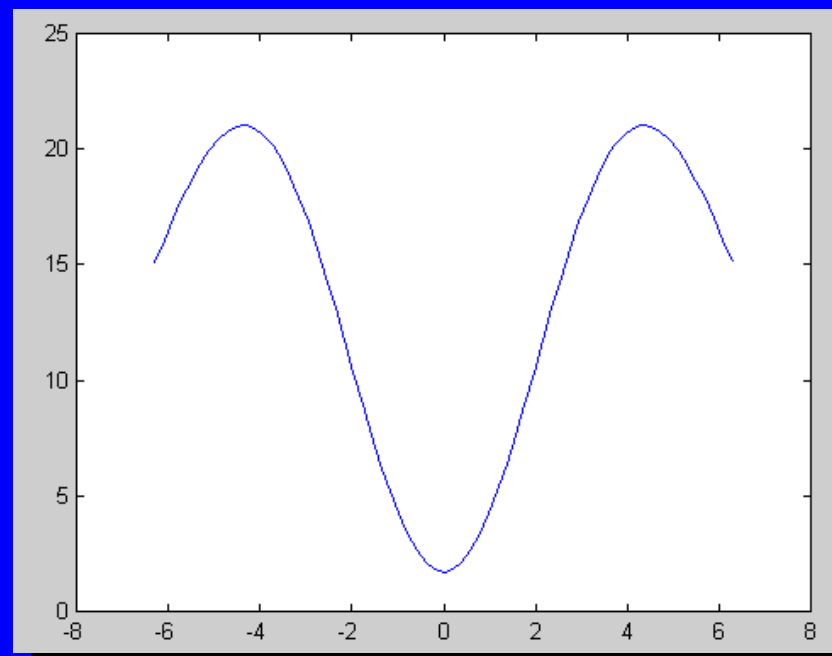
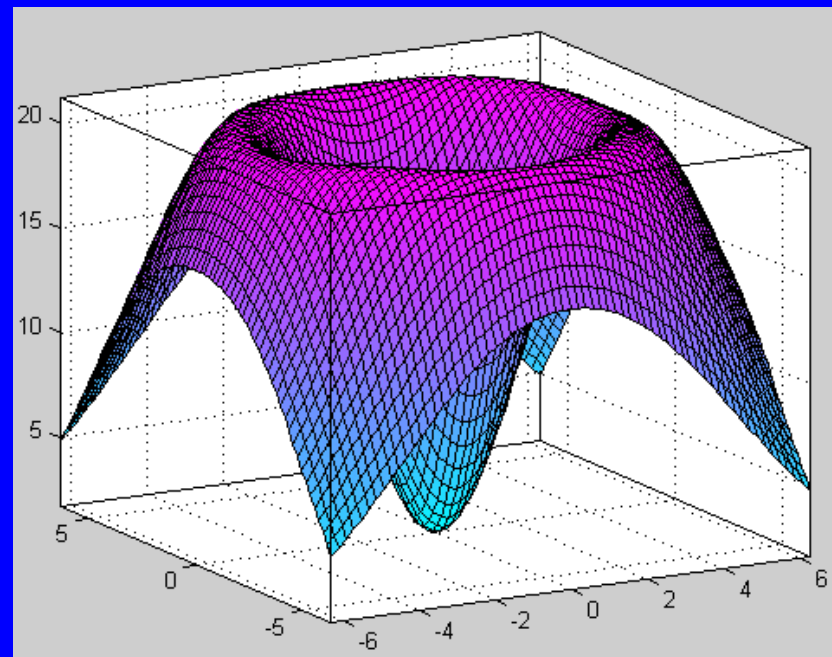
Where $\frac{1}{\pi\sigma^4}$ normalizes the sum of filter coefficients to 1, and σ controls the width of the main lobe.



Time domain



Fourier domain



LOG filter (Laplacian of Gaussian)

Generally,

$$g(x, y) = f(x, y) * h(x, y)$$

Here,

$$h(x, y) = \nabla^2 G(x, y)$$

which gives

$$g(x, y) = \nabla^2 G(x, y) * f(x, y)$$

It can be shown that:

$$\{\nabla^2 G(x, y)\} * f(x, y) = \nabla^2 \{G(x, y) * f(x, y)\}$$

This is equivalent to LP-filtering by Gaussian followed by HP-filtering by Laplacian.