MOMENTS

Definition

Given a 2D continuous function f(x,y), we define the moment of order (p+q) by the relation:

$$m_{p,q} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^q f(x,y)\, dx\, dy \qquad p,q=0,1,2...$$

In the discrete case:

$$m_{p,q} = \sum_x \sum_y x^p y^q f(x,y)$$

Central moments

The central moments are defined as:

$$\mu_{p,q} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\bar{x})^p(y-\bar{y})^q f(x,y)$$

$$\text{where } \bar{x} = \frac{m_{1,0}}{m_{0,0}} \text{ and } \bar{y} = \frac{m_{0,1}}{m_{0,0}}$$

Discrete case:

$$\mu_{p,q} = \sum_x \sum_y (x-\bar{x})^p(y-\bar{y})^q f(x,y)$$

The central moments are invariant under translation.

Scale change

Under scale change, $x' = \alpha x$, $y' = \alpha y$, the moments of $f(\alpha x, \alpha y)$ change to:

$$\mu'_{p,q} = \mu_{p,q} / \alpha^{p+q+2}$$

The normalized moments, defined as:

$$\eta_{p,q} = \frac{\mu'_{p,q}}{\left(\mu'_{0,0}\right)^{\gamma}} \quad \gamma = (p+q+2)/2$$

are then invariant to size change.

Rotation and reflexion

Under a linear coordinate transformation,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

it is possible to find certain polynomials of $\mu_{p,q}$ that remain unchanged under the transformation. For example, some moments invariant with respect to rotation (that is, $\alpha=\delta=\cos\theta$, $\beta=-\gamma=\sin\theta$) and reflexion ($\alpha=-\delta=\cos\theta$, $\beta=\gamma=\sin\theta$) are given as follows.

For $p+q=1$:

$$\Phi_0 = \mu_{0,1} = \mu_{1,0} = 0 \text{ (always invariant)}$$

For $p+q=2$:

$$\Phi_1 = \mu_{2,0} + \mu_{0,2}$$

$$\Phi_2 = \left(\mu_{2,0} - \mu_{0,2}\right)^2 + 4\mu_{1,1}^2$$

For $p+q=3$:

$$\Phi_3 = \left(\mu_{3,0} - 3\mu_{1,2}\right)^2 + \left(\mu_{0,3} - 3\mu_{2,1}\right)^2$$

$$\Phi_4 = \left(\mu_{3,0} + \mu_{1,2}\right)^2 + \left(\mu_{0,3} + \mu_{2,1}\right)^2$$

$$\Phi_5 = \left(\mu_{3,0} - 3\mu_{1,2}\right)\left(\mu_{3,0} + \mu_{1,2}\right)\left[\left(\mu_{3,0} + \mu_{1,2}\right)^2 - 3\left(\mu_{0,3} + \mu_{2,1}\right)^2\right]$$
$$+ \left(\mu_{0,3} - 3\mu_{2,1}\right)\left(\mu_{0,3} + \mu_{2,1}\right)\left[\left(\mu_{0,3} + \mu_{2,1}\right)^2 - 3\left(\mu_{3,0} + \mu_{1,2}\right)^2\right]$$

$$\Phi_6 = \left(\mu_{2,0} - \mu_{0,2}\right)\left[\left(\mu_{3,0} + \mu_{1,2}\right)^2 - \left(\mu_{0,3} + \mu_{2,1}\right)^2\right] + 4\mu_{1,1}\left(\mu_{3,0} + \mu_{1,2}\right)\left(\mu_{0,3} + \mu_{2,1}\right)$$